# Introduction to Graph Theory

## Hall’s Marriage Theorem

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Love is an interesting topic. It may be either too dangerous or too healthy for one’s mental health; and love is evil, spell it back, I will show you... However, the question is, is it possible to match each person in a community where everyone is happy with their choice? In other words, in which contiditions we can match everyone without excluding their criterias. Our attempt will be unterstanding these sort of problems.

In order to make easier to understanding, we consider two people, one girl and one boy. Say $K_1$ and $E_1$, respectively. If the girl $K_1$ is looking for a blonde and if $E_1$ is blonde, the match is trivial. If $E_1$ is not blonde, then a proper match is impossible. Therefore, it is not possible to make a suitable match for every community $\{K_1, E_1\}$. This has been an easy counter example for the problem, moreover, it is still not an answer to the main questions. For instance, we don’t know when we can match people and when not. Let’s give a more complicated example.

Assume that there are 6 toys to present 5 kids. We want to split these toys into kids where each kid has exactly one toy. Define the representations as below:

*Kids:* $\{C_1, C_2, C_3, C_4, C_5\}$

*Toys:* $\{O_1, O_2, O_3, O_4, O_5, O_6\}$

Let the table below show that which kid demands which toy:

| | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ |
|---|---|---|---|---|---|
| $O_1$ | 1 | | | 1 | |
| $O_2$ | | 1 | 1 | 1 | 1 |
| $O_3$ | 1 | | 1 | 1 | |
| $O_4$ | | 1 | | | |
| $O_5$ | | 1 | | | |
| $O_6$ | | 1 | | | |

As you can see in example, only option for $C_5$ is to match with $O_2$. In this case, $C_3$ must be matched with the toy $O_3$ . Then $C_4$ takes $O_1$. You must have seen what is going on here: Since we have already given the toys $O_1$ and $O_3$, we can’t match $C_1$ with any of these. Let’s try to understand in which cases we could have made a suitable match:

- If each kid demanded different toys.
- If each kid demanded every toy.
- If each kid demanded a number of toys between 1-5 where everyone demands different numbers of toys.

These examples go on... It may take forever, therefore we need to make better. Let’s give a definition, firstly:

**Definition:** Let $S$ be a collection of finite sets. $S$ itself may be infinite or repeat elements. For every $t \in T$, we mean a **transversal** by a bijection $f: T \to S$ where $t \in f(t)$.

Transversal gives us a choice function in reverse perspective. Since the function$f$ is a bijection, there exists $f^{-1}: S \to T$ and for every $\beta \in S$, $f^{-1}(\beta) \in \beta$. This is exactly a choice function.

**The Marriage Theorem:** For every subfamily $R$ of $S$, if the inequality $|R| \leq |\cup_{X \in R} X|$ is satisfied, then $S$ has at least one transversal.

The condition represented by the inequality is called marriage condition. So, we can say that $S$ has a transversal if and only if $S$ satisfies the marriage condition [1].

## An Example of Marriage

Let's say there is a case with $n$ women and $n$ men where everyone demands opposite gender. Here, choices of individuals will be one sided. In other words, women choose men according to their criteria and every man accepts to marry every woman. In addition, everyone can marry exactly one person. Then we can apply marriage theorem if the following statement occurs:

If in any group of women, the total number of men who are acceptable to at least one of the women in the group is greater than or equal to the size of the group

## Proof of Theorem

We need to know some fundamental definitons and theorems of posets in order to prove the main idea.

**Partially Ordered Set:** $S$ be a set with relation $\preccurlyeq$ . Suppose the set satisfies following statements:

- $\forall s \in S\ (s \preccurlyeq s)$ (Reflectivity)
- $\forall s, t, u \in S(s \preccurlyeq t \wedge t \preccurlyeq u \Rightarrow s \preccurlyeq u)$ (Transitivity)
- $\forall s, t \in S(s \preccurlyeq t \wedge t \preccurlyeq d \Rightarrow s = t)$ (Anti-Symmetry)

We call a poset (or Partially Ordered Set) such a set.

It's not need to be **comparable** each pair of elements in a poset. In another words, the statement

$$\forall s, t \in S\ (s \preccurlyeq t \vee t \preccurlyeq s)$$

need not to hold.

**Example 2.1:** Let $S = \{1, 2, 3, 4, 6, 12\}$ and define the order relation $n \preccurlyeq m$ if and only if $n$ divides $m$. This is a partial order relation and the elements 2 and 3 are not comparable.

**Definition:** Let $\preccurlyeq$ be a partial order relation over $S$. Let $Z \subseteq S$ be a subset. If every pair of elements of $Z$ is comparable, it is called **chain.** If $Z = S$ then the relation $\preccurlyeq$ is a **total order relation.**

For example the set $Z = \{1,2,4,12\}$ is a chain in (Example 2.1).

**Definition:** Let $A \subseteq S$. If $A$ has no comparable element pairs, then $A$ is called an **antichain**.

Again, in Example 2.1, $A = \{2,3\}$ is an anti-chain. We will use Dilworth's Theorem in order to prove Hall's Marriage Theorem. Dilworth's Theorem relates chains and antichains. Before that, let's try to construct a proof idea for marriage theorem.

Let's say the marriage condition holds for the family $R$. Let $\{X_1, X_2, \dots, X_n\} \subseteq R$ and for $x_j \in X_i$, let $A = \{x_1, \dots, x_k, X_1, \dots, X_n\}$. For every $i \in \{1,2, \dots, n\}$ and $j \in \{1,2, \dots, k\}$ define the relation $x_j \in X_i \Longrightarrow x_j \preccurlyeq X_i$. Moreover, for $i \neq j$ say $x_i$ and $x_j$ not to be comparable. Say the same for $X_i$ and $X_j$ , as well. The sets $R_{ij} = \{x_j, X_i\}$ represent a chain over the relation $\preccurlyeq$ , moreover, this chain is maximal. In addition, for every proper pair of $i, j$ the family of sets $R_{ij} = \{x_j, X_i\}$ cover $A$. In other words, $A \subseteq \bigcup_{i,j} R_{ij}$. On the other hand, the sets $\{x_1, x_2, \dots, x_k\}$ and$\{X_1, X_2, \dots, X_n\}$ are maximal antichains. It's left to the reader to see that. By marriage condition, the set $\{x_1, x_2, \dots, x_k\}$ has the maximum cardinality among maximal antichains.

This is all what we can do for now. We need Dilworth's Theorem in order to continue to prove.

**Dilworth's Theorem:** Let $S$ be a finite poset. Let $S$ has a maximal sized antichain $Y$ .Let $C$ be a minimal chain cover of $S$. Then the equality $|Y| = |C|$ holds.

We have already seen that each $R_{ij}$ is a chain. Moreover, these chains form a minimal cover for $A$. Here is why, it can be easily seen that the sets $\{x_j\}$ do not cover $A$. Hence, we need to add the sets $\{X_i\}$ into the covering. However, this family of sets is not a minimal cover. Because the sets $\{x_j, X_i\}$ form a less sized covering. There are no chain covers other than these cases. Denote $C_{min}$ such a cover. This results $|C_{min}| = k$. That means there is function such that

$\exists f: \{x_1, x_2, \dots x_k\} \to C_{min}$ ($f$ is bijection)

This function, defines a transversal over $R$. This completes the proof. Now, all we need to do is to prove Dilworth's Theorem.

## Proof of Dilworth's Theorem

There is a similiar theorem to the Dilworth's Theorem which is sort of a dual statement.

**Mirsky's Theorem**: Let $S$ be a finite poset. Let $Z$ be a maximal sized chain of $S$ . Let $C_{min}$ be a minimal sized antichain cover of $S$ . Then the equality $|Z| = |C_{min}|$ holds.

<u>Proof:</u> For any antichain; since there are no two comparable elements. Thus the every antichain cover is larger then $|Z|$. Therefore, $|C_{min}| \geq |Z|$ holds.

Define $f: S \to \mathbb{N}$ so that $f(s)$ represents the cardinality of the maximal chain where the largest element is $s$ . If $f(s) = f(t)$ and $t \neq s$ ,then the elements $t$ and $s$ are not contained by the same chain. Therefore, they are not comparable. This concludes by for any Bu da $n \in \mathbb{N}$ , the set $f^{-1}(n) = \{s\colon f(s) = n\}$ form a antichain. Moreover, the sets $f^{-1}(1), f^{-1}(2), \dots, f^{-1}(|Z|)$ cover $S$ . Therefore, $|Z| \geq |C_{min}|$ holds. This completes the proof.

<u>Proof (Dilworth's Theorem):</u> Let $S$ be a finite poset. $S$ has a maximal sized antichain and $C_Z$ be the minimal sized chain cover. We will use proof by induction on $|S|$.

If $|S| = 1$ then the set has only one cover and one antichain. There is nothing to prove in this case.

Suppose the theorem holds for every $|S| < n$ . If any two elements are not comparable, the sets $\{s\}$ form the only chain cover and since $\cup_{s \in S} = S$ holds, we have a minimal chain cover. Moreover, since no element pairs are comparable, we have the equality $S = Y$ which completes the proof in this case. Now, suppose $S$ has at least two comparable elements. Since $S$ is finite, there is a minimum and maximum comparable elements of $S$ . Call them $m$ and $M$ respectively. Let $T = S - \{m, M\}$. If $T$ has a antichain with at most $|Y| - 1$ cardinality, we can find a covering with $|Y| - 1$ elements by induction. Denote that covering as $C_T$ . The sets $C_T \cup \{m, M\}$ is a covering of $S$ with $|Y|$ elements. Theorem is proved in this case.

Now, suppose that $T$ has a maximal antichain with size $|Y|$ (Denote as $A$ ). Let's make following equations.

$$S^+ = \{x \in S\colon x \succcurlyeq a,\ \exists a \in A\}$$

$$S^- = \{x \in S\colon x \preccurlyeq a,\ \exists a \in A\}$$

Since $A$ is maximal, $S^+ \cup S^- = S$. Let's try to understand how the set $S^+ \cap S^-$ looks like. If $x \in S^+ \cap S^-$ then there is $a \in A$ where $a \preccurlyeq x$ and there is $b \in A$ where $x \preccurlyeq b$ . Thus, the relation $a \preccurlyeq x \preccurlyeq b$ holds but since $A$ is an antichain, the condition $a \preccurlyeq b$ holds if and only if $a = b$ . Therefore must have $x = a = b \in A$ . Hence, $S^+ \cap S^- = A$.

Since $m, M \notin A$ , (Recall that elements $m, M$ form a chain themselves) $m \notin S^+$and $M \notin S^-$. We can say that the sets $S^-$and $S^+$are proper subsets of $S$ . By induction hypothesis, these sets have an antichain covering of $|Y|$, furthermore each set in the covering contains exactly one element of $A$ . Denote the sets in the covering which contains $a$ as $C_a^-$ ve $C_a^+$. Furthermore, the sets in the form $C_a^- \cup \{a\} \cup C_a^+$ cover $S$ . Mind that each set in this covering expands the sets $S^+$and $S^-$, also mind that the number of the chain remains same.

This completes the proof by induction.
■

This has been a long proof but it is worth. Now we can really complete the proof of Hall's Marriage Theorem

**Corollary:** By the power of mathematics and Dilworth's Theorem, Hall's Marriage Theorem has been finally proved. ■

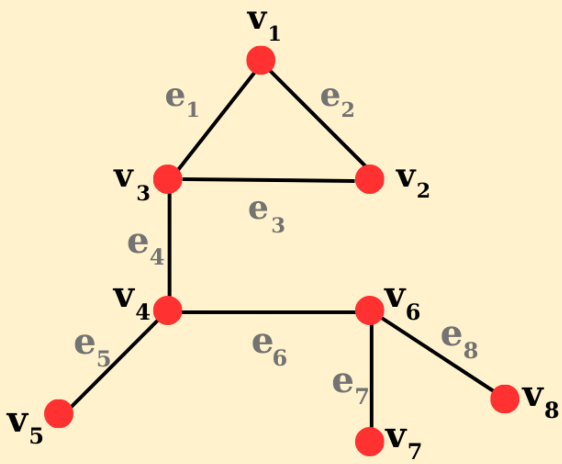
[2]

## Applications

### Bipartite Graphs

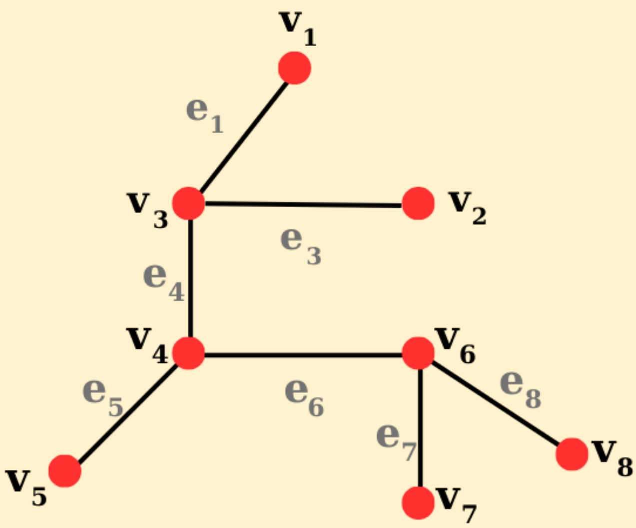
In this section, we will be working on *simple graphs*.

Now, we travel to the world of graphs! A **bipartite graph** is a graph where the vertices can be divided into two partitions $V_1$ and $V_2$ such that no two edges in the same set are adjacent. A **matching** on a graph is a choice of edges without common vertices. A matching *covers* a vertex set $V$ if each vertex has an endpoint in the edges of matching.




For example, the set of edges $\{e_3, e_5, e_7\}$ form a matching on the diagram above. And that matching covers the set of vertices $\{v_2, v_3, v_4, v_5, v_6, v_7\}$. The following statement gives us an information about how the marriage theorem can be used on matchings of a bipartite graph.

The role of Marriage Theorem here is if we have a bipartite graph with partition $V_1$ and $V_2$, there is a matching that covers $V_1$ if and only if for every subset $W$ of $V_1$, the number of edges in the graph with endpoints in $W$ is greater than or equals to $|W|$.

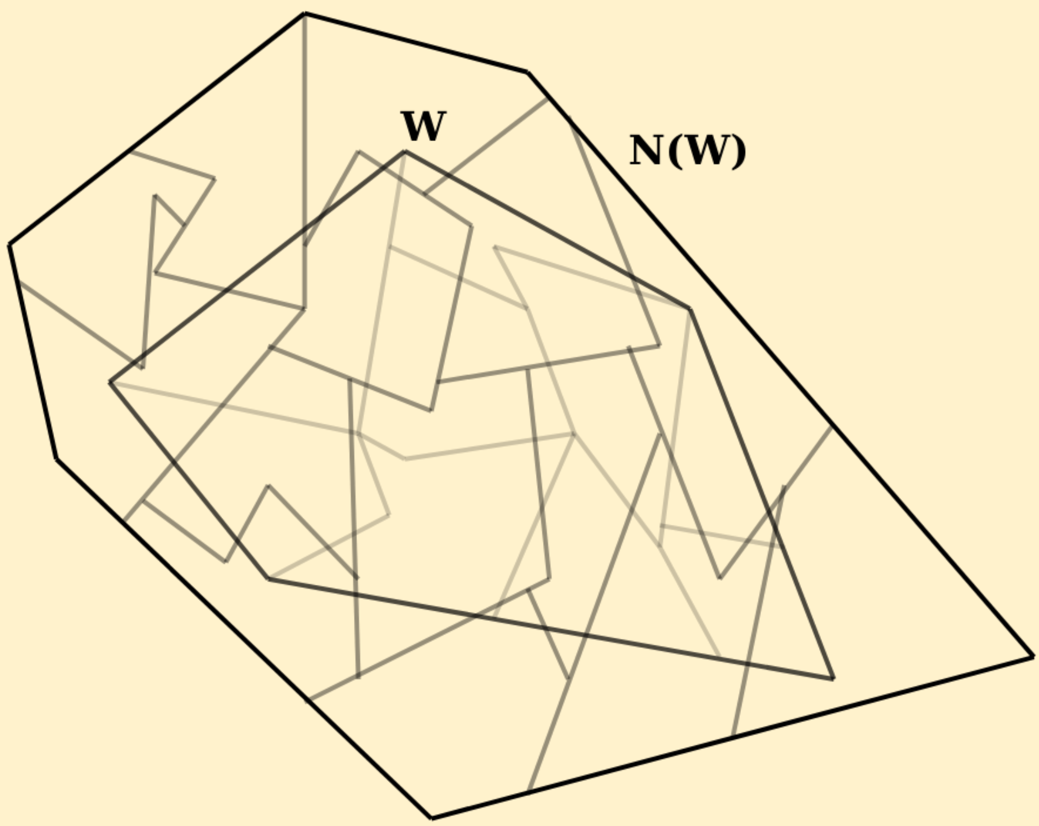


Follow the diagram above. See that this is a bipartite graph with $V_1 = \{v_1, v_2, v_4, v_7, v_8\}$, $V_2 = \{v_3, v_5, v_6\}$. There is no proper matching according to the marriage theorem. To see that, take $\{e_1, e_3\}$ and $W = \{v_1, v_2, v_3\}$. Since $2 \ngtr |W| = 3$, marriage condition does not hold [3].

**Problem (Kazakhstan 2003):** Suppose we are given two square sheets of paper which are divided into 2003 distinct polygons where each polygon has area 1. Locate the papers so that both papers are directly on top of each other. Show that we can place 2003 pins on the pieces of paper so that all 4006 polygons have been pierced.

First of all, we need to change this problem into a graph problem. Consider each polygon on the papers are vertices and each pin on the papers are edges between these vertices. Let $V_1$ be the set of vertices on one of the papers and let $V_2$ be the set of vertices of the other paper. By the assumption of problem, there are no adjacent vertices in $V_1$, or $V_2$. Thus, the graph is bipartite. Now, only thing we have to do is to show that the marriage condition holds.

Let $W$ be any subset of $V_1$. Now, draw the smallest $N(W) \subseteq V_2$ such that $N(W)$ contains the projection of $W$ on $V_2$.



The subset $W$ has $|W|$ polygon inside. Since each polygon has area 1 , the subset has area $|W|$. Also, it's neighborhood have area $|N(W)|$. Since projection of $W$ is bounded by $N(W)$, we can say that the inequality $|W| \leq |N(W)|$ holds. This completes the solution since we have already ended up with marriage condition. Hence, by the marriage theorem, there is a matching.

## Latin Squares

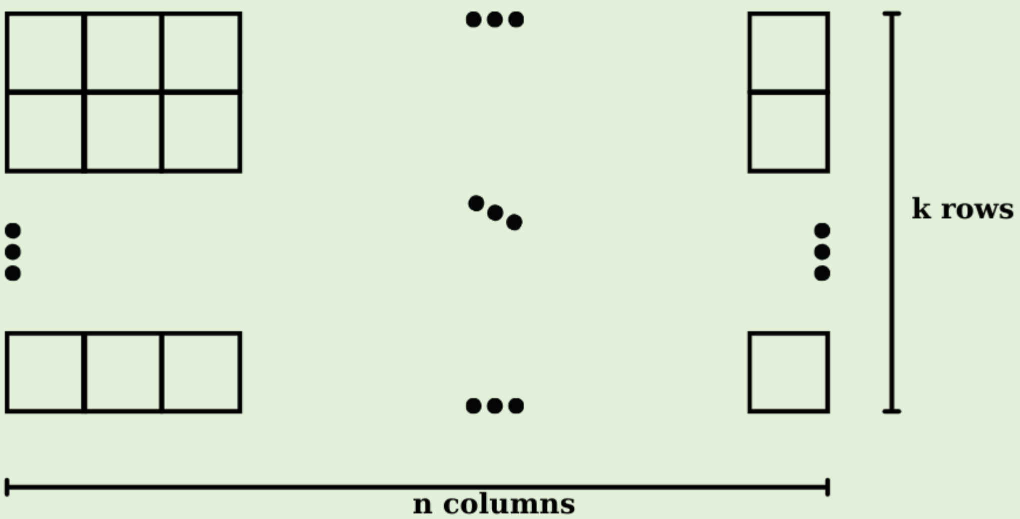
We have another problem related to bipartite graphs. Before we introduce the problem, we need a few definitions.

**Definition:** A **Latin Square** is a $n \times n$ square where each square is filled by the numbers $\{1,2,\dots,n\}$ such that there are no repeating elements in the same row or column.

Sudoku is a Latin square, for example.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 5 | 3 | 4 | 6 | 7 | 8 | 9 | 1 | 2 |
| 6 | 7 | 2 | 1 | 9 | 5 | 3 | 4 | 8 |
| 1 | 9 | 8 | 3 | 4 | 2 | 5 | 6 | 7 |
| 8 | 5 | 9 | 7 | 6 | 1 | 4 | 2 | 3 |
| 4 | 2 | 6 | 8 | 5 | 3 | 7 | 9 | 1 |
| 7 | 1 | 3 | 9 | 2 | 4 | 8 | 5 | 6 |
| 9 | 6 | 1 | 5 | 3 | 7 | 2 | 8 | 4 |
| 2 | 8 | 7 | 4 | 1 | 9 | 6 | 3 | 5 |
| 3 | 4 | 5 | 2 | 8 | 6 | 1 | 7 | 9 |

**Definition:** A **Latin Rectangle** is a $n \times n$ latin square that is completely filled in $k < n$ rows whereas it is empty in $n - k$ rows.



Now, the main question comes out.

**Problem:** Can we extend every Latin Rectangle to a Latin Square.

The answer is yes. Suppose we have $k \times n$ Latin rectangle. For the first step, we will extend this rectangle to a $(k+1) \times n$ rectangle. Then the result follows by induction. We need to construct a bipartite graph in order to solve the problem. Consider a graph with $2n$ vertices. We choose partition $V_1, V_2$ such that $|V_1| = |V_2| = n$ ,$V_1$ includes the $n$ columns of the rectangle and $V_2 = \{1, \dots, n\}$. We construct an edge between $V_i - j$ if $j \notin V_i$. This is a bipartite graph which can be easily seen by definition. Observe that since each column contains exactly one element of $V_2$ and there are $k$ rows, there are $n-k$ remaining option to put into a new row for any element of $V_2$. Hence, this is a $n-k$ regular bipartite graph. Now, we can check if the marriage condition holds. Let $W$ be any subset of $V_1$ with $p$ vertices. Since the garph is regular, there are $p(n-k)$ edges that are coming out from $W$ in total. Also, each vertex in $V_2$ has degree $n-k$ , so by the pigeonhole principle, there are at least $\frac{p(n-k)}{n-k} = p$ neighbors of edges. We get the result that $p \leq p(n-k)$, thus there is a matching for our graph by the marriage theorem. [4]

## References

[1] Anderson, I. (2002). *Combinatorics of finite sets*. Courier Corporation.

[2] Dilworth, R. P. (1987). A decomposition theorem for partially ordered sets. *Classic papers in combinatorics*, 139-144.

[3] West, D. B. (2001). *Introduction to graph theory* (Vol. 2). Upper Saddle River: Prentice hall.

[4] Quines, C. J. (2017). *Hall's marriage theorem.*