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DEPARTMENT OF MATHEMATICS

ALGEBRAIC STRUCTURES ON MUSIC THEORY

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List of Symbols

$\sharp$	Sharp Symbol
$\flat$	Flat Symbol
VH	The Set of Vertical-Horizontal Transformations
MT	The Set of Makam Transformations
M	The Set of Minor and Major Triads
$M_{b,c}$	The Set of Triads In The Form $\{a, a+b, a+c\}$ and $\{a, a+(c-b)+c\}$
TI	The Set of Transpositions and Inversions over $M_{b,c}$
PLR	The Set of P, L and R Transformations over $M_{b,c}$
NT	The Set of All Notes
$\mathbb{Z}$	The Set of Integers
$\frac{\mathbb{Z}}{12\mathbb{Z}}$	Quotient Group of $\mathbb{Z}$ over $12\mathbb{Z}$
$C_G(H)$	Centralizer of H over Group G
$Sym(X)$, S_X	Symmetry Group of X
$F(N)$	Frequency of the Note N
$cl(N)$	Conjugacy Class of Note N

Abstract

Western music operates with 12 tones, consisting of 7 diatonic (natural) and 5 chromatic (accidental) notes. When any three of these notes are played simultaneously, a triad is formed. Specifically, major and minor triads exhibit a D_{12} symmetry. This symmetry is generated by a T_n transformation of order 12—commonly referred to as transposition—and an I_n function of order 2—called inversion. Similarly, the group known as PLR is generated by the operations P, L, and R, each of order 2. These two groups are isomorphic and dual to each other in the context of major-minor triads. In this thesis, we generalize these transformations to apply them to other types of triads and adapt the same ideas to instruments used in Turkish music. Beyond this, we also explore how certain makam structures in Turkish music correspond to the group $\frac{\mathbb{Z}}{2\mathbb{Z}} \times \mathrm{Sym}(3)$.

Keywords: Notes, group, transformation, makam, comma

1.0 Introduction

The purpose of this article is to assert the relationship between group theory and music theory. Furthermore, we wish to determine the relations between Turkish folk music and Western music algebraically. This is difficult to work on since these topics seem to be irrelevant at first sight. The frequency (in terms of Hz) of a sound has already been determined; this is where the mathematics starts. For example, consider the sound of note C. If the frequency is increased precisely, we may obtain a note C, which is a higher C. Playing a melody one octave forward gives us the same melody but higher. Thus, the notes can be classified as equivalence classes. So, where is the group theory here? Firstly, examine the table of frequencies of notes.

Table 1.1: Frequency Table of Central Octave

Notes	Frequency (Hz)
Do	261.63
Do♯, Re♭	277.18
Re	293.66
Re♯, Mi♭	311.10
Mi	329.63
Fa	349.23
Fa♯, Sol♭	369.99
Sol	392.00
Sol♯, La♭	415.30
La	440.00
La♯, Si♭	466.16
Si	493.88
Do	523.50

It is observed that the ratio of the second Do and the first Do is exactly 2. Furthermore, it is seen that the ratio between two consecutive frequencies is exactly $2^{\frac{1}{12}}$. This fact is called as **equal tempered tuning**. From now on, we will denote the frequency of a note N as $F(N)$. Develop an approach to define equivalence classes of notes:

$$cl(N) = \{M : \exists n \in \mathbb{Z}, F(N) = 2^n F(M)\}$$

Choosing $n = 0$ in the definition gives the fact that $N \in cl(N)$. We need to prove that being an element of $cl(N)$ is an equivalence relation.

Table 1.2: Alphabetic Representations of Notes

Notes	Alphabet Representation
Do	C
Do♯, Re♭	C♯,D♭
Re	D
Re♯, Mi♭	D♯, E♭
Mi	E
Fa	F
Fa♯, Sol♭	F♯, G♭
Sol	G
Sol♯, La♭	G♯, A♭
La	A
La♯, Si♭	A♯, B♭
Si	B

Lemma 1. *Let N, M be two notes. Then either $cl(N) \cap cl(M) = \emptyset$ or $cl(N) = cl(M)$ is true.*

Proof: Suppose that the intersection is not empty. Say, $P \in cl(N) \cap cl(M)$. Then, for $n, m \in \mathbb{Z}$, we have $2^n F(N) = P$ and $2^m F(M) = P$. Then $2^{n-m} F(N) = F(M)$ holds, and since $n - m$ is an integer, it must be $N \in cl(M)$. If $N' \in cl(N)$ holds, then for an integer k, it must be $N' \in cl(M)$ holds since $2^k F(N') = F(N)$ is true together with $F(M) = 2^{n-m} F(N) = 2^{n-m} 2^k F(N') = 2^{n+k-m} F(N')$. However, this proves $cl(N) \subseteq cl(M)$. Without loss of generality, it can be shown that $cl(M) \subseteq cl(N)$. Hence, we have $cl(N) = cl(M)$ and this completes the proof. □

From the lemma, the corollary follows.

Corollary 1. *There are exactly 12 district sets $cl(N)$.*

From now on, we mean $cl(C)$ whenever we use C. From the corollary, there is a correspondence between the group $\mathbb{Z}/12\mathbb{Z}$ and the sets $cl(N)$. Now, we are so close to defining an algebra.

Note: For the rest of the article, we will use the alphabetic notation of notes, such as A, B, C, rather than La, Si, Do.

2.0 Groups

In this section, we attempt to embed the notes in the group. Denote the set of all note classes as NT. A bijection satisfying below can be defined:

$$f : \mathbb{Z}/12\mathbb{Z} \to NT,$$

$$f(\bar{0}) = A$$

. In addition, a binary operation of notes could have been written as $N+M := f(f^{-1}(N+M))$. In this case, we get $NT \cong \mathbb{Z}/12\mathbb{Z}$. Hence, the group $(NT, +)$ is cyclic with order 12. From now on, we can work with the group $\mathbb{Z}/12\mathbb{Z}$ instead of the set NT by using the definition above. In fact, the group $\mathbb{Z}/12\mathbb{Z}$ consists of equivalence classes of modulo 12. For an element $\bar{n} \in \mathbb{Z}$, we can write $\bar{n} = \{12k + n : k \in \mathbb{Z}\}$. This gives a similarity between the set $cl(N)$ and the set $\bar{n}$. Thus, defining the function $\alpha : cl(N) \to \bar{n}$ as such $\log_2(F(X)/F(N))$ gives us a bijection. This fact asserts that the cardinality of the set of all of the notes equals integers. One could even classify as isomorphic.
It could be any note chosen to be $\bar{0}$. This would not make a difference at our work. These kinds of definitions will not be important to us. We are going to examine the **triads**. A triad means when three notes are played at the same time.

Definition (Major Chord)**.** *Let $a \in \mathbb{Z}/12\mathbb{Z}$, then, the triad $\{a, a+4, a+7\}$ is called a major chord.*

The note a in this definition is called the **root** of the triad. For example, the triad $\{0, 4, 7\}$ represents **C major chord** whose root is C. In mathematics, there is no difference between the set $\{0, 4, 7\}$ and the set $\{4, 0, 7\}$. It is only considered the case of notes playing at the same time; thus, it does not need to include ordering. Hence, the root of the triad is invariant. For example, let's calculate the root of the chord $\{3, 8, 0\}$. It's easy to determine since there are only three options and the relations in the chord definition are obvious. 4 plus the elements 0 and 3 do not include the chord, so 8 is left to be the root. Since the equalities $8 + 4 = 0, 8 + 7 = 3$ are satisfied $8(Sol\sharp)$, is the root. We will denote major chords by capital letters. In this case, the chord $\{3, 8, 0\}$ is denoted as $G\sharp$.

Definition (Minor Chord)**.** *For $a \in \mathbb{Z}/12\mathbb{Z}$, the triad $\{a, a+3, a+7\}$ is called a minor chord.*

The note a in this definition is also called the **root** of the triad. Capital letters are used in order to denote major chords; now lower-case is used in order to denote minor chords. For example, the G minor chord is shown as $g\sharp = \{8, 11, 3\}$. Major and minor chords together are defined as a definite set, so some relations are asserted by defining some transformations over the set.

Definition. *There are 24 major and minor chords. Define the set M as containing all of these 24 chords.*
$M = \{\{a, a+4, a+7\}, \{A, A+3, A+7\} : a, A \in \mathbb{Z}/12\mathbb{Z}\}$

2.0.1 T and I Transformations

This section examines some special functions defined on the set M. One of these is called **transposition** and denoted as T_n. And the other is called **inversion** and denoted as I_n. But what is n here? There it is. Let $T_n : M \to M$ be defined by the rule:

$$T_n(\{a, b, c\}) = \{a+n, b+n, c+n\}$$

and let $I_n : M \to M$ be defined by the rule;

$$I_n(\{a, b, c\}) = \{-a+n, -b+n, -c+n\}.$$

The first thing that is observed is that there are exactly 24 distinct T_n and I_n transformations.

Lemma 2. *For every $n, k \in \mathbb{Z}$ such that $n \equiv k \pmod{12}$, $T_n = T_k$ and $I_n = I_k$ hold.*

Proof. By the definition of equivalence relation on n and k, there is an integer q such that $n = 12q + k$. Thus, we have:

$$T_n(\{A, B, C\} = \{A+n, B+n, C+n\}$$
$$= \{A+12q+k, B+12q+k, C+12q+k\}$$
$$= \{A+k, B+k, C+k\} = T_k(\{A, B, C\}).$$

One can calculate the same for I_n without loss of generality. Hence, we are done. □

Define the set $TI := \{T_n, I_n : n = 0, ..., 12\}$. It will be proven that this set is a group under composition in the following section.

2.0.1.1 Visualization of TI

Imagine a clock split into 12 equal intervals. Starting with the note A, a cyclic graph ordered from A to G is created as shown below.

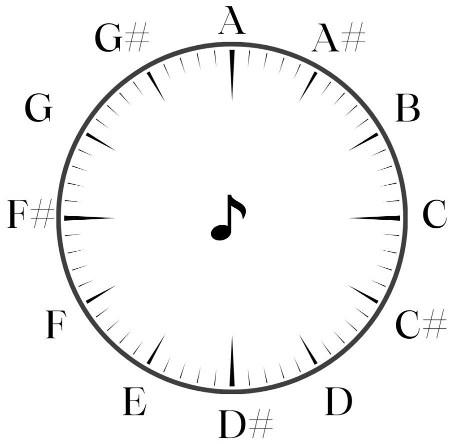


Figure 2.1: Note Visualization

Since the intervals are equal, the arc between consecutive notes is $\frac{\pi}{12}$. Consider the minor chord $\{0, 3, 7\}$. Mark the notes corresponding to 0, 3, and 7. Now a triad has been visualised:

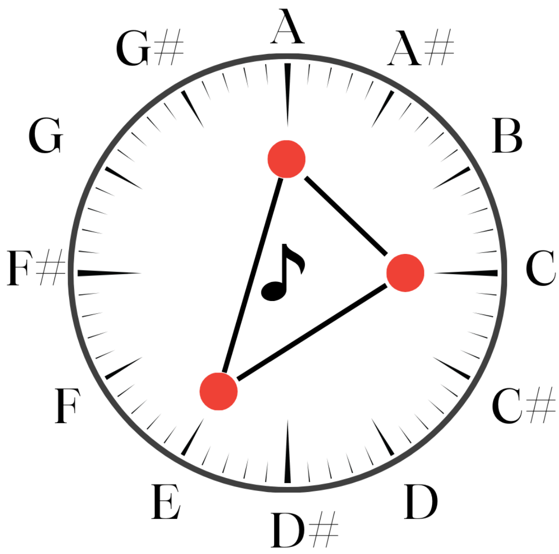


Figure 2.2: A-minor triad

This expresses a triangle with interior angles $\frac{\pi}{4}$, $\frac{\pi}{3}$ and $\frac{5\pi}{12}$. The transformation T_n translates every note by 3 units on the clock. Here, a new triad $\{a+n, a+3+n, a+7+n\}$ with the root $a+n$ is obtained. There are two conclusions:

1. Transposition T_n respects the parity (being major or minor),
2. Transformation T_n states a clockwise rotation by $\frac{n\pi}{6}$ around the centre of the circle.
3. Inversion I_n changes the parity
4. Transformation I_n states a reflection-rotation over the triangle.

Then, the set $\{T_n\}_{n=0,1,\ldots,12}$ is isomorphic to the set generated by clockwise rotations by $\pi/6$ of the triangle. In fact, instead of rotating the triangle, we can fix it and rotate the clock. We can draw a regular dodecagon whose vertices are notes and rotate it.

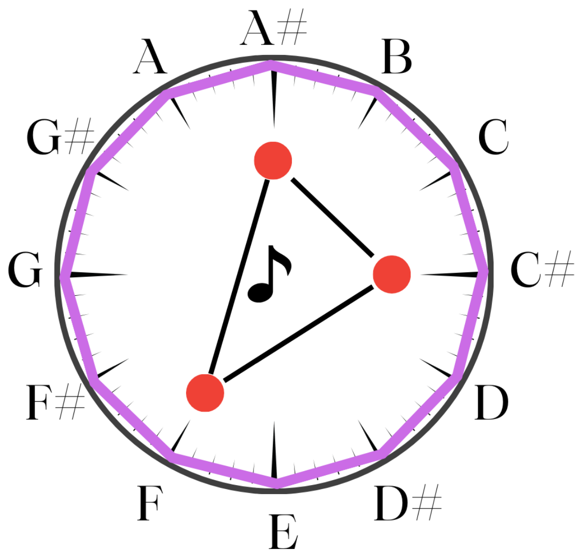
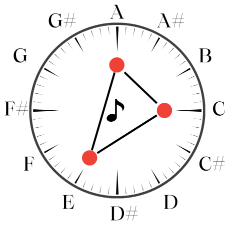
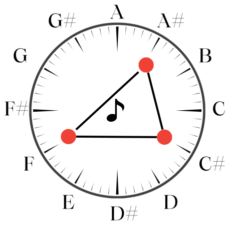


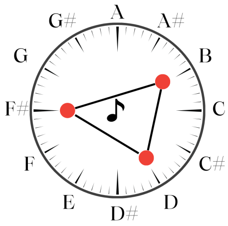
Figure 2.3: Dodecagon Representation of 12 Notes



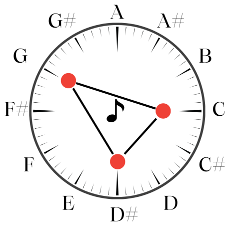
(a) a



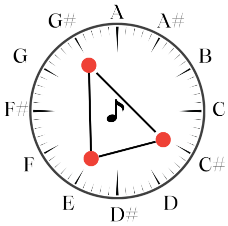
(b) b♭



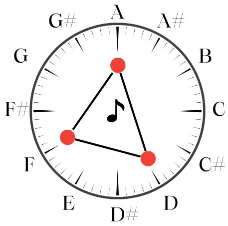
(c) b



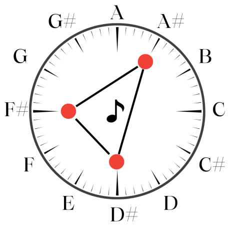
(d) c



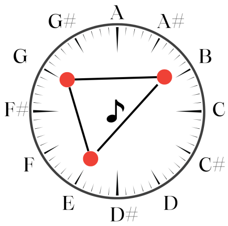
(e) c♯



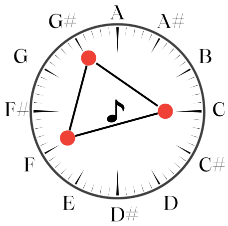
(f) d



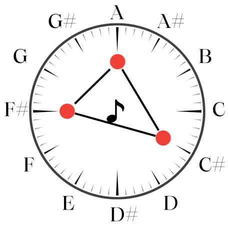
(g) d♯



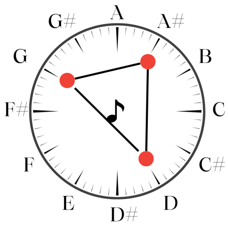
(h) e



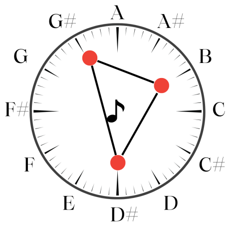
(i) f



(j) f♯



(k) g



(l) a♭

Figure 2.4: The Visualization Of Minor Chords

Similarly, the set $\{I_n\}_{n=0,\ldots,12}$ represents a reflection-rotation on both the triangle and dodecagon. From Figure 3, we can see the group TI is isomorphic to the dihedral group D_{12} .

2.0.1.2 The TI Set Algebraically

The purpose of this section is to prove that the set TI is a group without geometric intuitions.

By Lemma 3, there are exactly 12 T_n and 12 I_n transformations. Thus, the set TI has exactly 24 elements. Here is a useful lemma.

Lemma 3. *The following equalities hold:*

1. $T_m \circ \; T_n = T_{m+n \, (mod \, 12)}$
2. $T_m \circ I_n = I_{m+n \, (mod \, 12)}$
3. $I_m \circ T_n = I_{m-n \, (mod \, 12)}$
4. $I_m \circ I_n = T_{m-n \, (mod \, 12)}$

Proof. It is sufficient to do calculations below.

1.

$$\begin{aligned} T_m \circ T_n(\{A, B, C\}) &= T_m(T_n(\{A, B, C\})) \\ &= \; T_m(\{A+n, B+n, C+n\}) \\ &= \{(A+n)+m, (B+n)+m, (C+n)+m\} \\ &= T_{m+n \, (mod \, 12)}(\{A, B, C\}) \end{aligned}$$

2.

$$\begin{aligned} T_m \circ I_n(\{A, B, C\}) &= T_m(I_n(\{A, B, C\})) \\ &= \; T_m((\{-A+n, -B+n, -C+n\})) \\ &= \; \{(-A+n)+m, (-B+n)+m, (-C+n)+m\} \\ &= \; I_{n+m \, (mod \, 12)}(\{A, B, C\}) \end{aligned}$$

3.

$$\begin{aligned} I_m \circ \; T_n(\{A, B, C\}) &= I_m(T_n(\{A, B, C\})) \\ &= \; I_m(\{A+n, B+n, C+n\}) \\ &= \; \{-(A+n)+m, -(B+n)+m, -(C+n)+m\} \\ &= \; \{-A+(m-n), -B+(m-n), -C+(m-n)\} \\ &= \; I_{m-n \, (mod \, 12)}(\{A, B, C\}) \end{aligned}$$

4.

$$\begin{aligned} I_m \circ\ I_n(\{A,B,C\}) &= I_m(I_n(\{A,B,C\})) \\ &= I_m((\{-A+n, -B+n, -C+n\})) \\ &= \{-(-A+n)+m, -(-B+n)+m, -(-C+n)+m\} \\ &= \{A+(m-n), B+(m-n), C+(m-n)\} \\ &= T_{m-n\,(mod\,12)}(\{A,B,C\}) \end{aligned}$$

□

Due to this lemma, the composition TI is embedded in a sort of an addition. By this lemma, we can do the algebraic proof that the set TI is a group under composition.

Theorem 2. *TI is a group under composition of functions.*

Proof. By Lemma 3, the set TI is closed under composition, clearly. Thus, composition is a binary operation for TI. Furthermore, it is clear that the composition of functions is always associative. It is sufficient to show the existence of identity and inverse elements. By Lemma 3;

- $T_0\ \circ\ T_n = T_n\ \circ\ T_0 = T_{0+n} = T_n$
- $T_0\ \circ\ I_n = I_{0+n} = I_n = I_{n-0} = I_n\ \circ\ T_0$

hold for every element of TI. Then, T_0 is the identity element of the set. Again, by lemma 3;

- $T_n\ \circ\ T_{12-n} = T_0 = T_{12-n}\ \circ\ T_n$
- $I_n\ \circ\ I_n = T_{n-n} = T_0$

hold for every element of TI. Then every element has an inverse in TI. This proves the theorem.

□

The element T_1 has order 12, and all I_n's have order 2. Besides, the group is not abelian. From lemma 3, take $m = 4$, $n = 3$ for equalities $I_n \circ T_m = I_{m-n}$ and $T_m \circ I_n = I_{m+n}$ Since $I_1 \neq I_7$, the group is not abelian.

There is another detail about elements of TI. The T_n's respect the parity of chords, whereas I_n's do not. For example, a minor chord $\{x, x+3, x+7\}$ has an image under I_n:

$$\{-x+n, (-x-3)+n, (-x-7)+n\},$$

which is a major chord with root $-x-7+n$.

Now, examine the image of the major chord $\{x,\ x+4,\ x+7\}$ under I_n. The following equality

$$I_n(\{x,\ x+4,\ x+7\}) = \{-x+n,\ -(x+4)+n,\ -(x+7)+n\}$$

gives a minor chord with root $-x-7+n$.

It is obvious that T_n's do not change the parity. For example,

$$T_n(\{x,\ x+3,\ x+7\}) = \{x+n,\ x+3+n,\ x+7+n\}$$

is again a minor chord with root $x+n$. Similarly,

$$T_n(\{x,\ x+4,\ x+7\}) = \{x+n,\ x+4+n,\ x+7+n\}$$

is a major chord with a root of $x+n$.

2.0.2 P, L and R Transformations

There will be new transformations over M in this section. Those are called **parallel** (P), **leading tone exchange** (L) and **relative** (R).

Parity refers to the state of a chord being major or minor. For example, the chord $f = \{8, 11, 3\}$ has a minor parity, and the chord $F = \{8, 0, 3\}$ has a major parity. Two chords are said to be **parallel** if they have the same root but differ in parity. For example, preceding chords f and F are parallel. And the transformation P is defined to transpose the parallel chords. In other words:

$$P : M \to M, \quad P(\{A,\ A+4,\ A+7\}) = \{A,\ A+3,\ A+7\},$$
$$P(\{A,\ A+3,\ A+7\}) = \{A,\ A+4,\ A+7\}$$

Two chords are said to be relative if and only if they differ on parity and the minor one has a root 3 semitones below the major one's root. Then the definition of function R occurs as:

$$R : M \to M, \quad R(\{A,\ A+4,\ A+7\}) = \{A-3,\ A,\ A+4\},$$
$$R(\{A,\ A+3,\ A+7\}) = \{A+3,\ A+7,\ A+10\}$$

Finally, let $\{a, b, c\}$ be a triad with root a. The note $a - 1$ is called the **leading tone** of a. Take the major chord $\{a, a+4, a+7\}$. How could we write a chord with opposite parity who contains $a - 1$? The answer is$\{a+4, a+7, a-1 = a+11\}$! In backwards, take the minor chord $\{a, a+3, a+7\}$. Put $a+8$ instead of $a+7$. The triad $\{a+8, a, a+3\}$ is a major chord with root $a+8$. Then the transformation L is defined as:

$$L : M \to M, \quad L(\{A,\ A+4,\ A+7\}) = \{A+4,\ A+7,\ A-1\},$$
$$L(\{A,\ A+3,\ A+7\}) = \{A+8,\ A,\ A+3\}$$

These functions are bijections. One may try to imagine the visual representation of these transformations as we did for TI. It is common to use a system called **Tonnetz**. Before that, observe some useful meanings of P, L and R.

Suppose that we wish to transpose the chord $\{a, a+3, a+7\}$ from itself to a chord that fixes two of the notes in a triad. How many such transformations are there? Choose $a+3$ and $a+7$ fix. Then, our goal is to write a chord that contains both $a+3$ and $a+7$ but that is different from $\{a, a+3, a+7\}$. However, there is only one way to do that: changing the parity, as there are only two chords in M that contain both $a+3$ and $a+7$, which are $\{a, a+3, a+7\}$ and $\{a+3, a+7, a+10\}$. Thus, defining a transformation between these chords gives us the relative function R.

On the other hand, we could choose notes $a + 3$ and a to fix. Again, there are only two chords satisfying that, which are $\{a, a + 3, a + 7\}$ and $\{a + 8, a, a + 3\}$. A transformation that transposes these chords gives us the leading tone exchange L.

Finally, define a transformation fixing a and $a + 7$. There are again only two corresponding chords over M. Thus, the parallel function P is obtained this way.

Tonnetz Construct a path graph of notes in which the difference of adjacent nodes is 7. Imagine a system of such path graphs and connect each graph by two new graphs, where the difference of adjacent nodes is 4 and 3, respectively. There is the illustration of that graph below:

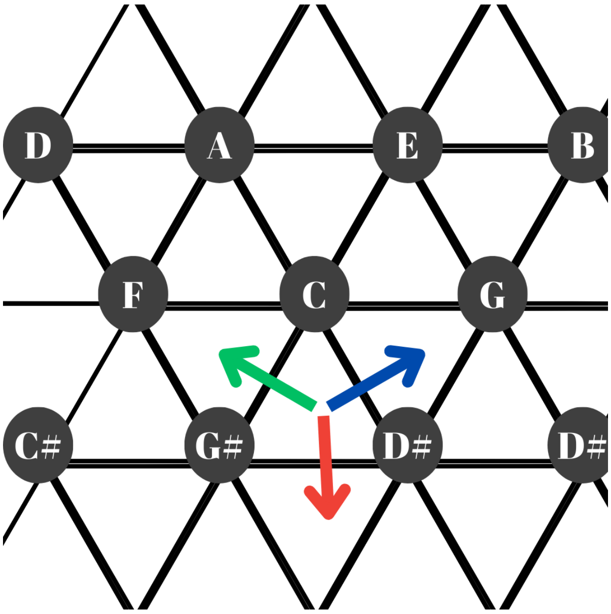


Figure 2.5: Oettingen / Riemann Tonnetz

Here, the edges of upside-down triangles form a minor chord, and plain triangles form a major chord. The green arrow represents relative (R); the blue arrow represents the leading tone exchange (L); the red arrow represents parallel (P). However, this representation of PLR is only made for a major triad. For minors, it becomes upside-down, as shown in Figure 5.

Also, there is the fact that each transformation PLR has order 2: $P^2 = R^2 = L^2 = i$. Tonnetz is important for us so that it is seen there is a group $PLR = \langle P, L, R \rangle$. By the Tonnetz diagram these transformations are closed under composition. Also, it is easily observed that every element in PLR has an inverse, as each of them has order 2. Thus, PLR is a group as desired.

Since $\langle \bar{7} \rangle = \frac{\mathbb{Z}}{12\mathbb{Z}}$, every note can be found in each row of the Tonnetz. Thus, only two of rows could have been sufficient to construct the diagram (see Figure 6).

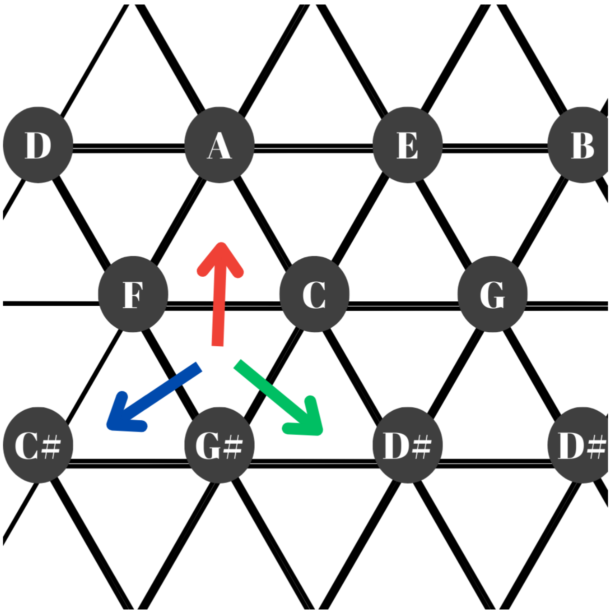


Figure 2.6: PLR Transformations For A Minor Chord

In this case, the function P is not needed in order to generate PLR. Since we have managed to list all of the chords in a shorter way, there is a sequence of compositions of L and R that is equal to P. Thus, write $PLR = \langle L, R \rangle$. So, how do we write P in terms of R and L? To do this, see that P only changes the middle element of a chord. For example, consider the chord $\{a, a+3, a+7\}$. A careful look at Figure 6, which we can use $L \circ R$ in order to switch to another minor chord which applies T_7 to the chord. Using the equality $a + 7 \times 4 = a + 28 = (a + 24) + 4 = a + 4$, three upside-down triangles right to the original place is the chord $\{a+9, a, a+4\}$. Applying L once again gives the chord $\{0, 4, 7\}$. Thus $P = R \circ (L \circ R)^3$. Now, it is needed to be sure that moving to the right and left is giving the same result over the Tonnetz. In other words, for each $(L \circ R)^n \circ R$, we must find a $(R \circ L)^n \circ R$ which is equal. To make things easier, say $RL = R \circ L$ and $LR = L \circ R$. Show that $\langle RL \rangle = \langle LR \rangle$.

Lemma 4. *Define $RL = R \circ L$ and $LR = L \circ R$, the equality $\langle RL \rangle = \langle LR \rangle$ holds.*

Observe that the calculation below is familiar. $R(L(\{a, a+3, a+7\})) = R(\{a+8, a, a+3\}) = \{a+5, a+8, a\}$

When one calculates the image of a minor chord under RL, it gives the same result as T_5 does. Similarly, $LR = T_7$ for minor chords. Take $\{a, b = a+3, c = a+7\}$ in order to show that:

$$L(R(\{a, a+3, a+7\})) = L(\{a+3, a+7, a+10\}) = \{a+7, a+10, a+2\}$$

Things change if we take a major chord; Take $\{a, b, c\}$ as $\{a, b = a+4, c = a+7\}$:

$$R(L(\{a, a+4, a+7\})) = R(\{a+4, a+7, a-1\}) = \{a+7, a-1, a+2\}$$

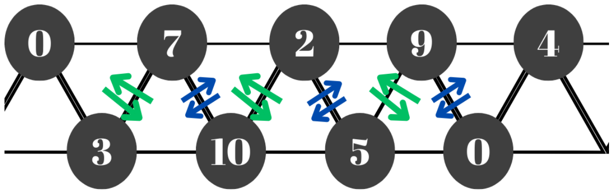


Figure 2.7: Tonnetz In Two Rows

$= \{a+7, a+11, a+2\}$, resulting in that $RL = T_7$. On the other hand,

$L(R(\{a, a+4, a+7\})) = L(\{a+9, a, a+4\}) = \{a+5, a+9, a\}$ resulting in that $LR = T_5$. It is interesting that the functions RL and LR acts oppositely on major and minor chords. But for all cases, the transformations T_5 and T_7 have order 12, so LR and RL are, regarless of parity. Hence, $Id_M = (RL)^{12} = (LR)^{12}$.

Proof of Lemma 4. Multiplying both sides by $(LR)^{-11}$ in $(RL)^{12} = (LR)^{12}$:

$$LR = (RL)^{12}(LR)^{-11} = (RL)^{12}(L \circ R)^{-11} = (RL)^{12}(R \circ L)^{11}$$
$$= Id_M \circ (RL)^{11} = (RL)^{11}.$$

Thus, $\langle RL \rangle \subseteq \langle LR \rangle$. Similarly, the relation $\langle LR \rangle \subseteq \langle RL \rangle$ holds. Hence, $\langle LR \rangle = \langle RL \rangle$ is proved. □

Remember that $R^2 = L^2 = Id_M$. In the proof of lemma 4, we used the facts that $R^{-1} = R$ and $L^{-1} = L$ without noticing. Consequently, PLR can be written as either $\langle LR, R \rangle$ or $\langle L, R \rangle$.

3.0 Group Isomorphisms

There are two different sets acting on M: TI and PLR. We already know that $TI \cong D_{12}$. It seems that the group PLR is also isomorphic to D_{12} since $PLR = \langle LR, R \rangle$ and $o(LR) = 12, o(R) = 2$. If this is the case, $PLR \cong TI$. That statement can be proved with the idea of matching T_1 by LR and I_0 by R. In other words, define the function

$\phi : PLR \to TI$ as $\phi(LR) = T_1$ and $\phi(R \circ (LR)^{-n}) = I_n$. Prove that the function ϕ is a group isomorphism, if it is possible.

Theorem 3 (Isomorphism)**.** *The function ϕ defined as above is a group isomorphism. Hence, $PLR \cong TI$.*

Proof. Since the functions LR and T_1 have the same order, for each n, there is a bijection between $(LR)^n$ and T_n. Furthermore, there are exactly 12 I_n transformations, so for each integer n, there is also a bijection between $R \circ (LR)^{-n}$ and I_n. This proves that ϕ is a bijection.

We should check if ϕ is a group homomorphism. To do that, for each $g, h \in PLR$ and each $x \in M$ we need to show that $\phi(g \circ h(x)) = (\phi(g) \circ \phi(h))(x)$. It is sufficient to examine the generators.

Let $g = LR$ and $h = R$. Suppose $x = \{a, b, c\} \in M$ is any element.

$$\begin{aligned} \phi(LR \circ R(x)) &= \phi((L \circ R \circ R)(x)) \\ &= \phi(L(x)) \qquad\qquad (R^2 = Id) \\ &= \phi(R \circ (LR)^{11}(x)) \\ &= I_1(x) \end{aligned} \tag{3.1}$$

Whereas,

$$\begin{aligned} (\phi(LR) \circ \phi(R))(x) &= (T_1 \circ I_0)(x) \quad (R = R \circ (LR)^0) \\ &= I_{1+0}(x) \qquad\quad (\textit{Lemma}\ 3) \\ &= I_1(x) \end{aligned} \tag{3.2}$$

since both sides give the same result, $\phi(g \circ h(x)) = (\phi(g) \circ \phi(h))(x)$ holds in this case.

Let $g = R$ and $h = LR$.

$$\begin{aligned} \phi(R \circ (LR)(x)) &= \phi(R \circ (LR)^{-11}(x)) \\ &= I_{11}(x) \end{aligned} \tag{3.3}$$

On the other side:

$$\begin{aligned} (\phi(R) \circ \phi(LR))(x) &= (\phi(R \circ (LR)^0) \circ \phi(LR))(x) \\ &= I_0 \circ T_1(x) \\ &= I_{0-1}(x) \qquad\qquad (\textit{Lemma}\ 3) \\ &= I_{11}(x) \end{aligned} \tag{3.4}$$

Two results are equal. Take $g = h = LR$.

$$\begin{aligned} \phi((LR) \circ (LR)(x)) &= \phi((LR)^2(x)) \\ &= T_2(x) \\ &= T_{1+1}(x) \\ &= (T_1 \circ T_1)(x) \\ &= \phi(LR) \circ \phi(LR) \end{aligned} \tag{3.5}$$

There is one case left: $g = h = R$,

$$\begin{aligned} \phi((R \circ R)(x)) &= \phi(Id(x)) \\ &= \phi((LR)^0(x)) \\ &= T_0(x) \\ &= Id(x) \\ &= (I_0 \circ I_0)(x) \\ &= (\phi(R \circ (LR)^{-0}) \circ \phi(R \circ (LR)^{-0}))(x) \\ &= (\phi(R \circ (LR)^0) \circ \phi(R \circ (LR)^0))(x) \\ &= (\phi(R) \circ \phi(R))(x) \end{aligned} \tag{3.6}$$

Thus, ϕ satisfies the homomorphism property for generators of PLR, so it satisfies for all elements. Thus, ϕ is a group homomorphism and we are done.

□

4.0 Duality

Since the groups TI and PLR consist of functions over M, there is group actions of these groups on M. This action is pretty natural. One may define group action of TI over M such that for each $x \in M$ and each $f \in TI$ as $f * x = f(x)$. Same definition could have done for PLR.

If $f = Id$ then it is clear that $f * x = f(x) = x$. Also, if $g, f \in TI$ then it is seen that $(g \circ f) * (x) = (g \circ f)(x) = g(f(x)) = g * (f(x)) = g * (f * x)$. Thus, M is a TI-set. Applying the same steps for PLR leads to that M is also a PLR-set. We have the following corollary, clearly.

Lemma 5. *The relations defined by rules $f * x = f(x)$, $* : TI \times M \to M$ and $* : PLR \times M \to M$ are group actions over M.*

Now, examine the orbits of these actions. Unfortunately, various of orbits can not be obtained here. Furthermore, take any $x \in M$, then $TIx = M$ holds. Here is the proof.

Lemma 6. *For every $x \in M$ it holds that every TI-orbit of x equals to M. In another words, for every $x \in M$ it holds that $TIx = M$.*

Proof. Here is what TI-orbit of x means:

$$TIx = \{f * x : \ f \in TI\}$$

For $f = T_0$ we have that $f(x) = x$. Since there are exactly 12 distinct T_n functions, acting over x gives at most 12 elements that have the same parity in TIx. Choose $n, m \in \{0, 1, ..., 11\}$ such that $n \neq m$. In order to make the equality $T_n(x) = T_m(x)$ hold, hence, $T_n(x) \circ T_{-m}(x) = T_m(x) \circ T_{-m}(x)$. By Lemma 3, it can be written that $T_{n-m}(x) = T_0(x) = x$. The last equality is true for every x, thus $n - m = 0$ and then $n = m$. Hence, there are exactly 12 elements that have the same parity with x in TI.

Check the functions I_n. Since the elements $I_0(x)$ and x have opposite parity, for every $n = 0, 1, ..., 11$, it holds that $I_n(x) \in TIx$ such distinct elements as we mentioned in the preceding paragraph. Thus, we had added at most 12 more elements in TIx. However, for $n, m \in \{0, 1, ..., 11\}$, the equality $I_n(x) = I_m(x)$ holds if and only if $I_n \circ I_{-m}(x) = I_m \circ I_{-m}(x)$ and this holds if and only if $T_{n-m}(x) = T_0(x)$. From these, $n = m$. Thus, those additional 12 elements are distinct.

By the definition of orbit, $TIx \subseteq M$. Since $|TIx| = |M| = 24$, then it is proved that $TIx = M$. □

In this proof, it is also mentioned that for each $x \in M$ and for distinct $f, g \in TI$ we have $f(x) \neq g(x)$. This gives an idea of what the stabilisers of x look like. We have $f(x) = x$ if and only if $f = Id_M$.

Lemma 7. *Let $x \in M$ and $TI_x = \{f \in TI : f * x = x\}$. Then the set TI_x has only 1 elements that is identity.*

Proof. We can use the orbit stabiliser theorem:

$$|TI_x| = \frac{|TI|}{|TIx|}.$$

This equality holds for every $x \in M$. By Lemma 6, $|TI| = |TIx| = 24$ thus it holds that $|TI_x| = 1$. Also, for every $x \in M$ since $Id_M(x) = x$ then $Id_M \in TI_x$. Hence, the proof is done. □

One can prove every detail that we have seen for group actions PLR. So, $|PLR_x| = 1$ which means the only element is $Id_M \in PLR_x$. This kind of action is called **free action**. Thus, the groups TI and PLR act freely on M.

Lemma 8. *TI acts on M transitively. We mean by that, for every pair of $x, y \in M$ there is an $f \in TI$ such that $f * x = y$.*

Proof. This lemma is an outright corollary of Lemma 6. Since $y \in M = TIx$ then there is an $f \in TI$ such that $f * x = y \in TIx = M$. □

TI and PLR act on M both freely and transitively. This kind of action is called **regular action**.

The road so far, the groups TI and PLR are isomorphic but not equal. These groups are not Abelian. However, one may ask, what do the centralisers of these groups look like? In fact, TI and PLR consists of elements of functions from M to M. Hence these groups are subgroups of $Sym(M)$ such that are isomorphic. In notations:

$$TI \leq Sym(M)$$

$$PLR \leq Sym(M)$$

$$TI \cong PLR$$

Thus, the elements of those two groups can be composited. Better than that, this composition is commutative.

Lemma 9. *Suppose $f \in TI$ and that $g \in PLR$, then $f \circ g = g \circ f$ holds.*

Proof. It is sufficient to do this proof for generators of groups. To do this, use the fact that $PLR = \langle LR, R \rangle$. For TI, take T_1 and I_0 as generators. In fact, $T_n = T_{1+\ldots+1} = (T_1)^n$ and $I_n = I_{0-(-n)} = I_0 \circ T_n$ by Lemma 3.

There are 4 cases which each case splits into 2 case:

1. Case 1 for minor chords

$$\begin{aligned} R \circ T_1(\{a, a+3, a+7\}) &= R(\{a+1, a+4, a+8\}) \\ &= \{a+4, a+8, a+11\} \\ &= T_1(\{a+3, a+7, a+10\}) \\ &= T_1 \circ R(\{a, a+3, a+7\}) \end{aligned}$$

Case 1 for major chords

$$\begin{aligned} R \circ T_1(\{a, a+4, a+7\}) &= R(\{a+1, a+5, a+9\}) \\ &= \{a+10, a+1, a+5\} \\ &= T_1(\{a+9, a, a+4\}) \\ &= T_1 \circ R(\{a, a+4, a+7\}) \end{aligned}$$

2. Case 2 for minor chords

$$\begin{aligned} LR \circ T_1(\{a, a+3, a+7\}) &= L \circ R \circ T_1 \\ &= L \circ T_1 \circ R \end{aligned}$$

Leave a pause here and see that for the rest of the proof, it is enough to show that L and T_1 are commutative.

$$\begin{aligned} L \circ T_1(\{a, a+3, a+7\}) &= L(\{a+1, a+4, a+8\}) \\ &= \{a+9, a+1, a+4\} \\ &= T_1(\{a+8, a, a+3\}) \\ &= T_1 \circ L(\{a, a+3, a+7\}) \end{aligned}$$

Case 2 for major chords:

$$\begin{aligned} L \circ T_1(\{a, a+4, a+7\}) &= L(\{a+1, a+5, a+8\}) \\ &= \{a+8, a+5, a\} \\ &= T_1(\{a+7, a+4, a+11\}) \\ &= T_1 \circ L(\{a, a+4, a+7\}) \end{aligned}$$

3. Case 3 for minor chords:

$$\begin{aligned} R \circ I_0(\{a, a+3, a+7\}) &= R(\{-a-7, -a-3, -a\}) \\ &= \{-a-10, -a-7, -a-3\} \\ &= I_0(\{a+3, a+7, a+10\}) \\ &= I_0 \circ R(\{a, a+3, a+7\}) \end{aligned}$$

Case 3 for major chords:

$$\begin{aligned} R \circ I_0(\{a, a+4, a+7\}) &= R(\{-a-7, -a-4, -a\}) \\ &= \{-a-4, -a, -a+3\} \\ &= I_0(\{a-3, a, a+4\}) \\ &= I_0 \circ R(\{a, a+3, a+7\}) \end{aligned}$$

4. Case 4 for minor chords:

$$\begin{aligned} L \circ I_0(\{a, a+3, a+7\}) &= L(\{-a-7, -a-3, -a\}) \\ &= \{-a-3, -a, -a+4\} \\ &= I_0(\{a-4, a, a+3\}) \\ &= I_0 \circ L(\{a, a+3, a+7\}) \end{aligned}$$

Case 4 for major chords:

$$\begin{aligned} L \circ I_0(\{a, a+4, a+7\}) &= L(\{-a-7, -a-4, -a\}) \\ &= \{-a+1, -a-7, -a-4\} \\ &= I_0(\{a+4, a+7, a-1\}) \\ &= I_0 \circ L(\{a, a+4, a+7\}) \end{aligned}$$

For all cases, the generators are commutative under function composition. Thus, any element in TI and any other element in PLR commutes. □

From all these, the groups PLR and TI are centralisers of each other in $Sym(M)$. Above all, these groups are dual.

Definition. *Let X be any nonempty set and let $H, K \leq Sym(X)$. If these subgroups acts regularly on X and they are centralizers of each other, so that $C_{Sym(X)}(H) = K$ and $C_{Sym(x)}(K) = H$, these groups are called **dual** of each other over X.*

There is one more fact that we need to show. One might say that the groups TI and PLR are subsets of corresponding centralisers; however, not necessarily equal to. The following theorem says that centralisers are equal.

Theorem 4 (Duality)**.** *It holds that* $PLR = C_{Sym(M)}(TI)$ *and* $TI = C_{Sym(M)}(PLR)$*. Moreover, the groups* PLR *and* TI*,* M *are dual over* M *since they act on* M *regularly.*

Proof. Without loss of generality, we will show that $PLR = C_{Sym(M)}(TI)$. First, remember the definition of centralizers of TI on $Sym(M)$:

$$C_{Sym(M)}(TI) = \{g \in Sym(M) : f \circ g = g \circ f, \forall f \in TI\}$$

It is clear that $PLR \subseteq C_{Sym(M)}(TI)$ by Lemma 9. The purpose is to prove the difference of these sets is empty, so $C_{Sym(M)}(TI) - PLR = \phi$. Suppose that $x \in M$. Examine the stabiliser of x on $C_{Sym(M)}(TI)$. For an $h \in C_{Sym(M)}(TI)$, suppose that $h(x) = x$. Then:

$$f(h(x)) = f(x)$$

$$h(f(x)) = f(h(x)) = f(x).$$

The first equality is clear. For the second, the fact that the elements h and f commute is used. Moreover, TI acts transitively, so for every $y \in M$ there is $g \in TI$ such that $y = g(x)$. Substituting y into the second equality gives:

$$h(y) = h(g(x)) = g(x).$$

Thus, for every $y \in M$, there is $h(y) = y$. In fact, h is the identity function: $(h = Id_M)$. By using the orbit stabiliser theorem, write that:

$$\begin{aligned} |C_{Sym(M)}(TI)x| &= \frac{|C_{Sym(M)}(TI)|}{|C_{Sym(M)}(TI)_x|} \\ &= |C_{Sym(M)}(TI)| \leq |M| \\ &= 24 \end{aligned}$$

The last equality came from that orbit $C_{Sym(M)}(TI)$ of x is a subset of M. Since the stabiliser and the group have the same cardinality, these are less than the cardinality of M . Furthermore, as $24 = |PLR| \leq |C_{Sym(M)}(TI)|$, it must be $24 = |C_{Sym(M)}(TI)|$. Thus we get $|C_{Sym(M)}(TI) - PLR| = 0$. Hence, the difference is empty. The proof has been completed. □

Finally, there is the corollary of duality of TI and PLR.[1]

5.0 Generalization of Structure on M

We have observed some transformations over the the set triads M who acts regularly on M and have duality over M. Is there any structures that satisfy these conditions? Our purpose in this section is to find such sets and groups.

The set M has two types of elements: major chords and minor chords. Any chord can be seen in the form $\{a, a+b, a+c\}$ for any $a \in \frac{\mathbb{Z}}{12\mathbb{Z}}$ and definite $b, c \in \frac{\mathbb{Z}}{12\mathbb{Z}}$. There is a relation $\{a,\ a+b,\ a+c\} \xleftrightarrow{\text{Parity}} \{a,\ a+(c-b),\ a+c\}$ between two parities. For minor chords, $b = 3, c = 7$.

Define a new set $M_{b,c} := \{\{a,\ a+b,\ a+c\},\ \{a,\ a+(b-c),\ a+c\} : \forall a \in \frac{\mathbb{Z}}{12\mathbb{Z}}\}$. Here, b and c are fixed elements of $\frac{\mathbb{Z}}{12\mathbb{Z}}$. Define the functions $M_{b,c} \to M_{b,c}$:

$T_1(\{a, b, c\}) = \{a+1, b+1, c+1\}$,

$I_0(\{a, b, c\}) = \{-c, -b, -a\}$,

$P : \{a, a+b, a+c\} \xleftrightarrow{\text{Parity}} \{a, a+(c-b), a+c\}$,

$R : \{a, a+b, a+c\} \xleftrightarrow{\text{Parity}} \{a+b, a+c, a+b+c\}$,

$L : \{a, a+b, a+c\} \xleftrightarrow{\text{Parity}} \{a+(-c+b), a, a+b\}$,

then let $PLR = \langle P, L, R\rangle$ and $TI = \langle T_1, I_0\rangle$. Can we generalize the results we obtained for M? Firstly, consider the set $M_{5,10}$. Then $P(\{0, 5, 10\}) = \{0, 5, 10\}$. Therefore, the action of PLR over $M_{5,10}$ is not free. Thus, it is not regular. Hence, we cannot talk about duality of the PLR and TI groups here. So we need a restriction of choice of b and c: $c \neq 2b$. In this condition, $c - b \neq b$ so that P changes elements.

Another point that we should take into account is that $PLR \cong TI$. In order for this work, we need the order of LR to be 12 and order of R to be 2. Since $2b \neq c$, the second condition is always satisfied. So, we should focus on the first condition. Calculate LR:

$$LR(\{a, a+b, a+c\}) = L(\{a+b, a+c, a+b+c\}) = \{a+c, a+b+c, a+2c\}$$

Thus, for the first-type elements; $LR = T_c$. For the second types:

$$LR(\{a, a+c-b, a+c\}) = L(\{a-b, a, a+c-b\}) = \{a-c, a-b, a\}$$

We have $LR = T_{-c}$ for the second-type elements. Since orders of c and $-c$ are equal, LR has order 12 if and only if order of c is 12. We can say that $c \in \{1, 5, 7, 11\}$.

By now, here are all the possibilities that b and c can take.

Table 5.1: Table of b's and c's

b	c
0	1
0	5
0	7
0	11
1	5
1	7
1	11
2	5
2	7
2	11
3	7
3	11
4	11
5	11

In fact, the set M is equal to $M_{3,7}$. For all b's and c's in the table, the transformations PLR and TI can be generalized over $M_{b,c}$ such that:

$$PLR \cong TI \cong D_{12}$$

and the groups PLR and TI are dual over $M_{b,c}$.

Note: The case $b = 0$, $\{a, a+b, a+c\}$ is a triad, but it is not a set with cardinality 3. We still play three notes, but two of them are equivalent. So, we can see them as triple pairs: $(a, a, a + c), (a, a + c, a + c)$. All definitions and corollaries that we stated in this section are preserved.

6.0 Introduction to Turkish Music

Turkish music seems so complicated compared to the Western music system, and it is so difficult to adapt an algebraic concept. Therefore, in this paper, we will work with the system of a special instrument: **Baglama**. In this section, we will construct the basics of this instrument. Similarly with guitar, it uses strings. It differs with the string numbers. There are 7 strings in total, where the strings are separated into 3 groups:

- **Group A**: It sounds the note A without touching any pitch. It is called **Top String**.
- **Group G**: It sounds the note G without touching any pitch. It is called **Middle String**.
- **Group D**: It sounds the note D without touching any pitch. It is called **Bottom String**.

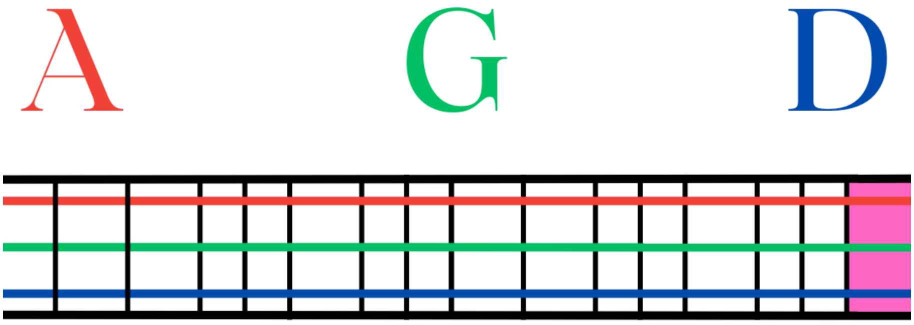


Figure 6.1: The String Groups of Baglama

In order to make sound, you can touch and press a specific pitch and play the string group. Here are some notes on the baglama:

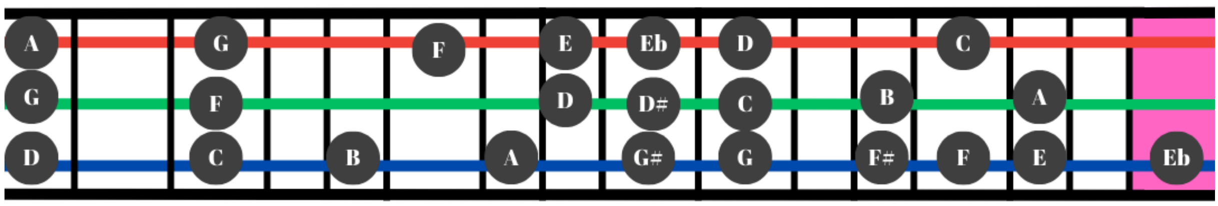


Figure 6.2: Notes

As you can see, there are some missing pitches that are not classified. These notes do have theoretical classifications, but they are rarely used in performance practice. We will assume their existence and work with them also since they are also parts of symmetry.

Classification of Notes In the first section, we have seen that a note is a sound of a specific frequency. Divide two consequent notes into nine equal intervals. One of those intervals is called a "**comma**" (koma in Turkish).

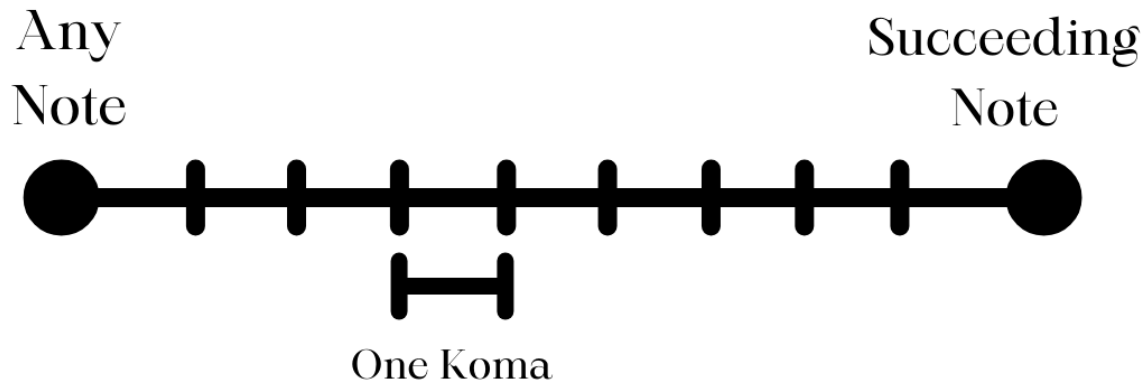


Figure 6.3: Visualization of Commas

Mind that two commas of two distinct note intervals do not have the equal interval.

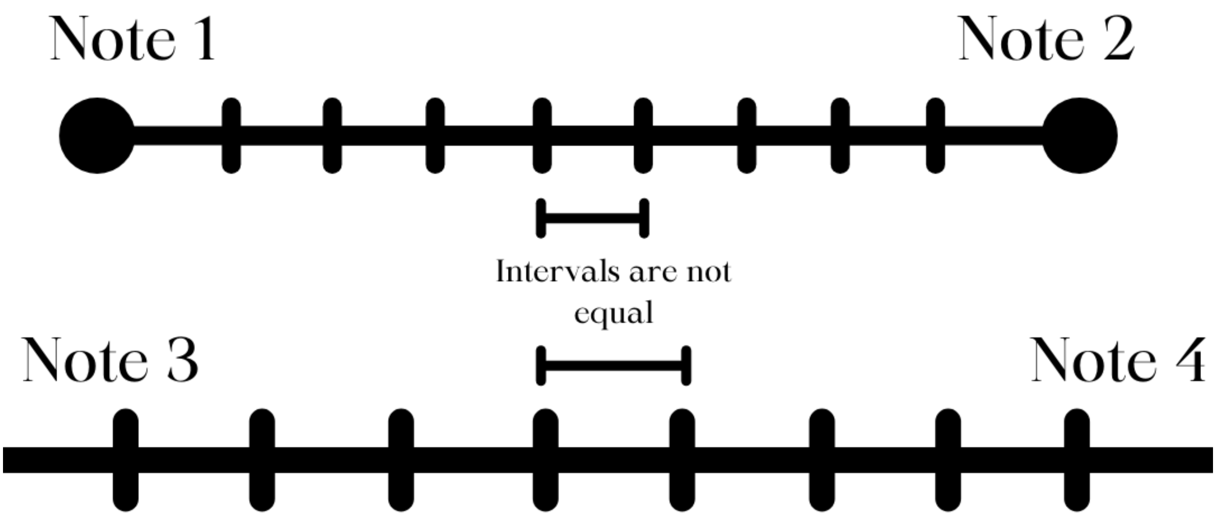


Figure 6.4: Intervals of Commas

However, we can still count how many commas are there in two distinct commas. For example, the notes A and B have 9 commas, the notes A and C have 18 commas, the notes A and A♯ have 4.5 commas.

Definition. *Let N_1 and N_2 be two notes. We will say that N_1 is congruent to N_2 if and only if there are 63 commas between them.*

In this system, we will work with 17 distinct note classes instead of 12. Realise that in Figure 8, two pitch intervals may not be equally divided. This is because it uses the comma system.

Here is the comma distribution of baglama.

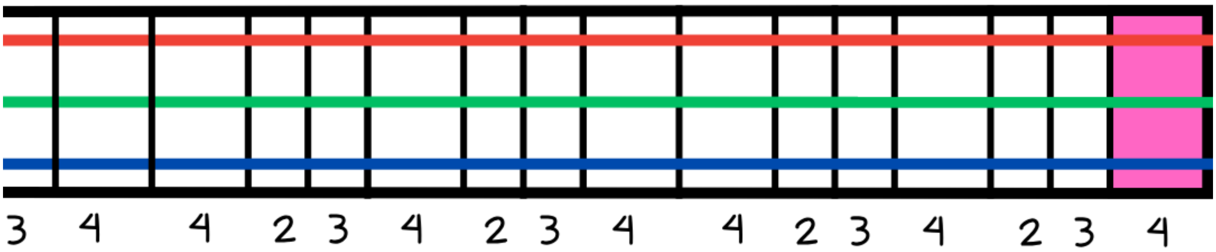


Figure 6.5: Comma Distribution of Baglama[3]

Now that we have introduced the comma-based pitch system, we will define special note groupings fundamental to Turkish makam music.

6.0.1 Makam System

In this subsection, our purpose is to understand some special structures of Turkish music[2].

Definition (Quadruple)**.** *Four notes that have 22 comma intervals in total is called a* ***total quadruple****.*

Quadruples start with a stop point. So, every total quadruple can be seen as (S, i_1, i_2, i_3) where S is a note and i_1, i_2, i_3 are positive integers such that $i_1 + i_2 + i_3 = 22$. For example, $(S, 4, 9, 9)$, $(S, 9, 4, 9)$, $(S, 9, 9, 4)$, $(S, 9, 8, 5)$, $(S, 5, 12, 5)$ are total quadruples.

Lemma 10. *Let S be any note and (S, i_1, i_2, i_3) be any total quadruple. Let $\sigma \in Sym(3)$ any permutation. Then $(S, i_{\sigma(1)}, i_{\sigma(2)}, i_{\sigma(3)})$ is a total quadruple.*

Proof. By the commutativity of integers, $22 = i_1 + i_2 + i_3 = i_{\sigma(1)} + i_{\sigma(2)} + i_{\sigma(3)}$. □

Definition (Quintuple)**.** *Five notes that have 31 comma intervals in total are called a* ***total quintuple****.*

Thus, every total quadruple can be represented as (S, i_1, i_2, i_3, i_4) where S is any note and $i_1 + i_2 + i_3 + i_4 = 31$. A similar corollary to lemma 10 is valid for quintuples.

Lemma 11. *Let S be any note and (S, i_1, i_2, i_3, i_4) be a total quintuple. For any $\sigma \in Sym(4)$, $(S, i_{\sigma(1)}, i_{\sigma(2)}, i_{\sigma(3)}, i_{\sigma(4)})$ is a total quintuple.*

The proof of Lemma 11 is analogous to that of Lemma 10.

Lemma 12. *Let S, M be arbitrary notes. Let (S, i_1, i_2, i_3) be a total quadruple. Then $(M, i_1, i_2, i_3, 9)$ is a total quintuple.*

Proof. Since $i_1 + i_2 + i_3 = 22$ is already satisfied, adding 9 to both sides yields to: $i_1 + i_2 + i_3 + 9 = 31$. □

For example, $(S, 4, 9, 9, 9)$ is a total quintuple by lemma 12.

Note: In practice, not every quadruple can be extended with 9 commas to form a valid quintuple in all makam constructions. The musical context may impose further constraints.

Definition. *A* ***simple makam*** *is a definite pair of total quadruple-quintuple.*

For example, a quadruple $(S, 4, 9, 9)$ and a quintuple $(S + 22, 4, 9, 9, 9)$ can be combined in a simple makam. It suits since the stop point of the first makam is the end point of another. Thus, we can denote this makam as $(S, 4, 9, 9, 4, 9, 9, 9)$.

Definition. *A transposed makam is a simple makam whose stop point is transposed.*

For example, let $(S, i_1, i_2, i_3, i_4, i_5, i_6, i_7)$ be a simple makam. Then $(M, i_1, i_2, i_3, i_4, i_5, i_6, i_7)$ is a transposed makam where $M \neq S$. One may define a map $\tau_n : (S, i_1, ...) \to (S + n, i_1, ...)$ to make a mathematical definition.

7.0 Algebraic Structures On Baglama

In this section, our goal is to assert algebraic relations over baglama.

7.0.1 Triads Over Baglama

Our first purpose is to construct a set X and groups G_1, G_2 such that $G_1 \cong G_2$ and they are dual over X. There are a variety of triads in baglama. We could choose $B = M$, $G_1 = TI$ and $G_2 = PLR$. However, this is trivial since we are already done with it. Fortunately, we can do something similar.

We will see triads as triples (a, b, c) instead of $\{a, b, c\}$. This is because the places of notes are important. The note a is played on the top string, whereas the note b is played on the middle string, and the note c is played on the bottom string. Furthermore, $a = b$ might be written this time. While doing all of these, we will work with 12 notes by removing unnecessary elements of our 17 notes.

Now, we should understand how triads work:

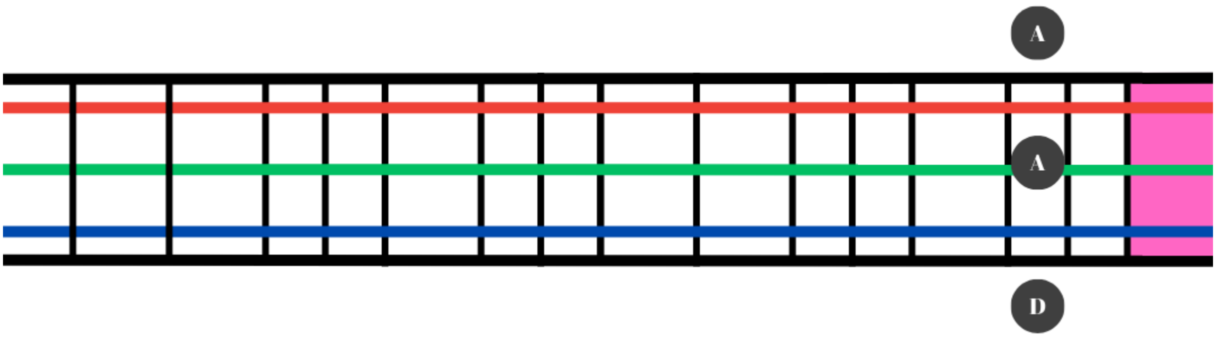


Figure 7.1: D-bare

1. Locate your finger to the indicated interval in Figure 13 of the middle string (green line).

2. Do not touch the top and bottom strings, so they will sound notes A and D, respectively.

3. Play all string groups.

This will sound like D in practice. Thus, this is a (A, A, D) triad. This triad has root D, not A. Triads are called bares ("Bare" or "Karar" in Turkish) in baglama, so we may also call them $D-$bare. That forms a triple of $(x+7, x+7, x)$. We would see that this is an element of $M_{0,7}$ if it were a set instead of a triple. Starting from this, we can define the set X.

$X := \{(x+7, x+7, x),\ (x+7, x, x) : x \in \frac{\mathbb{Z}}{12\mathbb{Z}}\}.$

7.0.1.1 TI group over X

We can define transpositions and inversions for B, but there will be a small difference from the original.

$$T_n : X \to X, \qquad T_n((a,b,c)) = (a+n, b+n, c+n)$$
$$I_0 : X \to X, \qquad I_0((a,b,c)) = (-c, -b, -a)$$

It can be easily seen that these definitions satisfy Lemma 3. Now, we can define the group.

$$TI := \langle T_1, I_0 \rangle$$

The generator T_1 has order 12 and the generator I_0 has order 2. Furthermore, the group has 24 elements. Observe that $TI \cong D_{12}$. To see that we visualise:

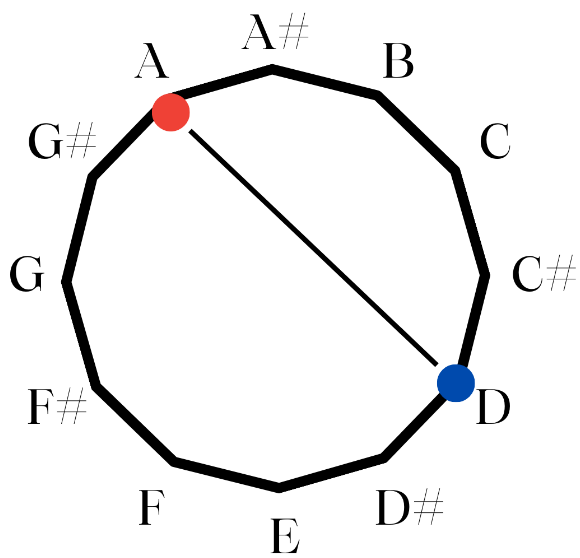


Figure 7.2: A Dodecagon Whose Vertices Are Notes

In Figure 14, the stick located at the middle of the dodecagon represents our triad. The red vertice represents the double note in the triad, and the blue vertice represents the single note. Rotating the dodecagon clockwise by $\frac{\pi}{6}$ is equivalent to applying T_1 to the stick. Reflecting the dodecagon with respect to the perpendicular bisector of the vertices $B-C$ is equivalent to applying $T_3 \circ I_0$ to the stick. Therefore, $TI \cong D_{12}$.

7.0.1.2 LR Group Over X

This group is similar to PLR. However, since at least two of the coordinates of the elements in X are already equal, it must be $P = L$. Thus, it is sufficient to define L and R. Let $L, R : X \to X$ be such that:

$$L : (x+7, x+7, x) \longleftrightarrow (x+7, x, x)$$

$$R : (x+7, x+7, x) \longleftrightarrow (x+2, x+7, x+7)$$

Now, we know that LR has order 12 and R has order 2 according to Section 5. Set LR to $\langle L, R\rangle$. Now, we have $LR \cong TI \cong D_{12}$. Consequently, the triple (X, TI, LR) satisfies the desired properties.

7.0.2 Makam Transformations

We shall observe makam structures on the baglama board. There are three types of pitches on the baglama: 4-comma pitches, 3-comma pitches and 2-comma pitches. The sequence of these pitches can be represented as

$$(4,3,2)(4)(3,2,4)(4,3,2)(4)(3,2,4)(4,3,2)$$

What is the purpose of the parentheses in the sequence? Consider the first three terms. The parentheses $(4,3,2)$ is 9 commas in total. Now, we shall represent the sequence by total commas of group of parentheses:

$$9,4,9,9,4,9,9$$

This sequence forms a simple makam. There are various ways to see how this makam is distributed. See Figure 15. The groupings (4, 9, 9), (9, 9, 4), (9, 4, 9) are total quadruples with the stop indicated in pink. The groupings (9, 4, 9, 9), (9, 9, 9, 4) and (9, 9, 4, 9) are total quintuples. Let P_{B_4}, P_{B_5} be the bottom stops of quadruples and quintuples be respectively, P_{M_4}, P_{M_5} the middle stops and P_{T_4}, P_{T_5} be the top stops in the same way.

On the one hand, define the following elements:

- $m_1 = (P_{B_4}, 9, 4, 9)$
- $m_2 = (P_{B_5}, 9, 4, 9, 9)$
- $m_3 = (P_{M_4}, 9, 4, 9)$
- $m_4 = (P_{M_5}, 9, 4, 9, 9)$
- $m_5 = (P_{T_4}, 9, 4, 9)$
- $m_6 = (P_{T_5}, 9, 4, 9, 9)$

And let $VH = \{m_1, m_2, m_3, m_4, m_5, m_6\}$.

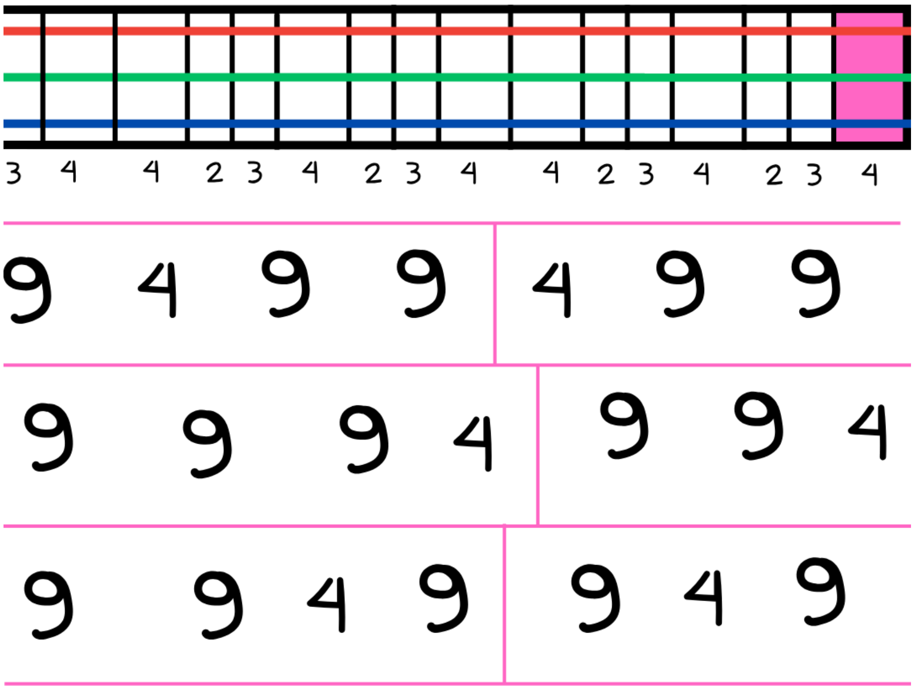


Figure 7.3: Comma Distribution of Makam

On the other hand, let P be any stop point and define the following elements:

- $M_1 = (P, 9, 4, 9)$
- $M_2 = (P, 9, 9, 4)$
- $M_3 = (P, 4, 9, 9)$
- $M_5 = (P, 9, 4, 9, 9)$
- $M_4 = (P, 9, 9, 4, 9)$
- $M_6 = (P, 4, 9, 9, 9)$

and let $MT = \{M_1, M_2, M_3, M_4, M_5, M_6\}$

7.0.2.1 Transformations Over VH

On the board of the baglama, the player sometimes plays 2 notes simultaneously, sometimes 3 notes, and sometimes they play only 1 note. Player might pass 3 notes from 2 notes or three notes from another.

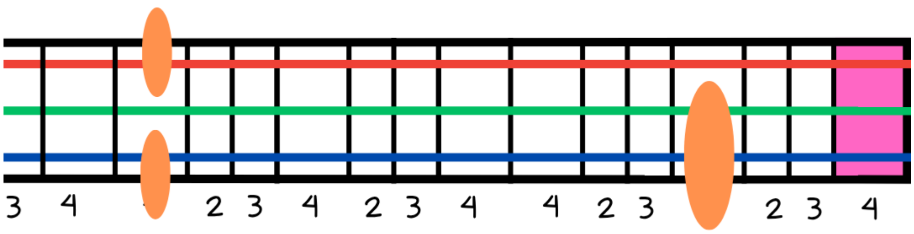


Figure 7.4: Horizontal Moves

In Figure 16, imagine that you are playing the little orange spotted notes. Then you want to pass the big orange spot. You were playing the top and bottom strings and then you passed to the middle and bottom strings. Thus, we can construct the permutation $(TM)(B)$. This is an idea of defining some functions over VH which is isomorphic to $Sym(3)$. This is a vertical transition. Let,

$$V_\sigma : VH \longrightarrow VH$$

by the rule,

$$V_\sigma((P_{X_c}, 9, 4, 9)) = (P_{\sigma(X)_c}, 9, 4, 9)$$

where $\sigma \in Sym\{T, B, M\}$ and $c \in \{4, 5\}$.

Lemma 13. $\langle V_\sigma \rangle_{\sigma \in Sym\{T,B,M\}} \cong Sym(3)$.

Proof. Let $f : Sym(3) \longrightarrow \langle V_\sigma \rangle_{\sigma \in Sym\{T,B,M\}}$ by the rule $f(\sigma) = V_\sigma$. Let $\sigma, \beta \in Sym(3)$ arbitrary permutations. Then,

$$\begin{aligned} f(\sigma\beta)(P_{X_c}, 9, 4, 9) &= V_{\sigma\beta}((P_{X_c}, 9, 4, 9)) \\ &= (P_{\sigma\beta(X)_c}, 9, 4, 9) \\ &= V_\sigma((P_{\beta(X_c)}, 9, 4, 9)) \\ &= V_\sigma \circ V_\beta((P_{X_c}, 9, 4, 9)) \\ &= f(\sigma) \circ f(\beta)((P_{X_c}, 9, 4, 9)) \end{aligned}$$

Thus, f is a group homomorphism. Suppose that $f(\sigma)(P_{X_c}, 9, 4, 9) = (P_{X_c}, 9, 4, 9)$ for every $X \in \{T, B, M\}$. This is satisfied if and only if $\sigma(X) = X$ for every X, so σ is identity function. Hence, the kernel of f is trivial. Therefore, f is one-to-one. f is onto since all V_σ's are determined by σ's. Thus, f is a group isomorphism. □

Let $H : VH \longrightarrow VH$ by the rule:

$$H((P_{X_4}, ...)) = (P_{X_5}, ...)$$

$$H(P_{X_5}, ...) = (P_{X_4}, ...)$$

The group $\langle H \rangle$ is isomorphic to $\frac{\mathbb{Z}}{2\mathbb{Z}}$ since the order of H is 2. Let $G_{VH} := \langle H, V_\sigma \rangle$. The transformation H acts on makams horizontally, so we called the set as VH.

Corollary 5. $G_{VH} \cong Sym(3) \times \frac{\mathbb{Z}}{2\mathbb{Z}}$.

7.0.2.2 Transformations Over MT

Define $F : MT \longrightarrow MT$ by the rule:

- $F(P, Q_4) = (P, Q_5)$
- $F(P, Q_5) = (P, Q_4)$

where $Q_4 = (i_1, i_2, i_3)$ and $Q_5 = (i_1, i_2, i_3, 9)$. For example, $F(P, 9, 4, 9) = (P, 9, 4, 9, 9)$. It is easily observed that $F^2 = Id$.

Define $F_\sigma : MT \longrightarrow MT$ by the rule:

$$F_\sigma((P, Q)) = (P, \sigma(Q))$$

where σ is a permutation that takes quadruples to quadruples and quintuples to quintuples. To be more elaborative, consider Figure 17. Imagine black dots as total quadruples and gray dots as total quintuples. The function F_σ takes one of vertical quadruple-quintuple pair to another and the function F takes quintuples to associated quadruples.

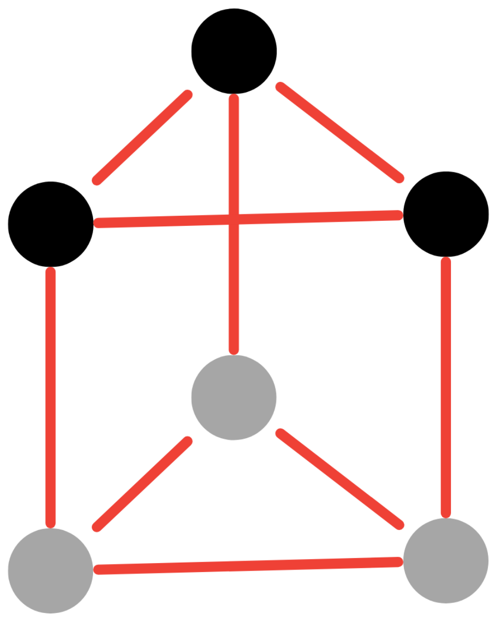


Figure 7.5: Graph of MT

Therefore, we can say that $\langle F, F_\sigma \rangle$ is isomorphic to automorphisms of the graph above. The graph in Figure 17 is regular since all nodes have degree 3.

Theorem 6. *The following are satisfied:*

- $\langle F \rangle \cong \frac{\mathbb{Z}}{2\mathbb{Z}}$
- $\langle F_\sigma \rangle \cong Sym(3)$
- $\langle F, F_\sigma \rangle \cong Sym(3) \times \frac{\mathbb{Z}}{2\mathbb{Z}}$

The proof of Theorem 6 is similar to Lemma 13. Let $G_{MT} := \langle F, F_\sigma \rangle$.

Corollary 7. $G_{MT} \cong G_{VH}$.

7.0.2.3 Extensions Of G_{MT} and G_{VH}

Define the set

$$K := \{(P, Q_n, Q_m) : (n,m) = (4,5) \text{ or } (n,m) = (5,4),\ \ Q_4 = (i_1, i_2, i_3)$$
$$\text{s.t. } i_{1,2,3} \in \{4,9\},\ i_1 + i_2 + i_3 = 22,\ \ Q_5 = (i_1, i_2, i_3, 9),$$
$$P \in \{P_{B_4}, P_{M_4}, P_{T_4}, P_{B_5}, P_{M_5}, P_{T_5}\}\}$$

as set of makams. The group G_{VH} preserves the comma digits and changes the stop point whereas the group G_{MT} preserves the stop point and changes the structure of makam. Now, we extend elements of G_{VH} and G_{MT}. Let $G = G_{VH} \cup G_{MT}$ and $f \in G$. By the following rule,

$$f((P, Q_n, Q_m)) := \begin{cases} (f(P), Q_n, Q_m) & f \in G_{VH} \\ (P, f(Q_n), f(Q_m) & f \in G_{MT} \end{cases}$$

the function f can be extended to the set K.

Lemma 14. *Let $f \in G_{VH}$ and $g \in G_{MT}$. Then $f \circ g = g \circ f$*

Proof. Choose any $(P, Q_n, Q_m) \in K$.

$$\begin{aligned}
(f \circ g)((P,\ Q_n,\ Q_m)) &= f(g((P,\ Q_n,\ Q_m))) \\
&= f((P,\ g(Q_n),\ g(Q_m))) \\
&= (f(P), g\ (Q_n), g\ (Q_m)) \\
&= g((f(P),\ Q_n,\ Q_m)) \\
&= g(f((P,\ Q_n,\ Q_m))) \\
&= (g \circ f)((P,\ Q_n,\ Q_m))
\end{aligned}$$

Since the element (P, Q_n, Q_m) is arbitrary, we are done. □

Another observation on K is that action of G on K is not free. For example, both permutations $(23), Id \in Sym(3)$ fix 1. Thus, there can be found $V_\sigma \in G_{VH}$ such that $V_\sigma(P,\ Q_n,\ Q_m) = (\sigma(P),\ Q_n,\ Q_m) = (P,\ Q_n,\ Q_m)$ where $\sigma \neq Id$.

However, the action of G on K is still transitive. To see that, let $(P,\ Q_n,\ Q_m), (S,\ Q'_n,\ Q'_m) \in K$. There is always an element $f \in G_{VH}$ such that $f(P) = S$. Consider the graph automorphisms of Figure 17. There can be found an automorphism such takes one particular vertice to another one. Thus, there can be found $g \in G_{MT}$ such that $f(Q_n,\ Q_m) = (Q'_n,\ Q'_m)$. This shows that action of G on K is transitive.

Corollary 8. *Let* $* : G \times K \longrightarrow K$ *be a group action defined by the rule* $f * x = f(x)$*. Then the following are satisfied:*

- $G_{MT} \subseteq C(G_{VH})$ *and* $G_{VH} \subseteq C(G_{MT})$*. (By Lemma 14)*
- *The action* $*$ *is transitive.*

8.0 Results And Discussion

This paper introduced the symmetries of the notes. These transformations, dualities, or isomorphisms make it possible to play music in different styles. In chapter 7.0, the symmetry between makams of baglama is observed. The symmetry shows us that any music over baglama can be played based on various of makam changes determined by the group $S_3 \times \mathbb{Z}/12\mathbb{Z}$. There are two main playing style over baglama: Playing by plectrum or playing by fingers. The standart playing style is by plectrum. However, it may be possible using fingers as capo and change makams over keyboard of baglama due to the symmetry. That symmetry is constrained for simplicity. For example, one may define 17 notes over baglama and study on $S_3 \times \frac{\mathbb{Z}}{17\mathbb{Z}}$. To be more clear, there would be 17 horizontal transformations and vertical transformations isomorphic to S_3. This paper aimed to introduce group structure over Turkish music and various of groups can be studied on as explained.

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